

INEQUALITIES VIA s -CONVEXITY AND \log -CONVEXITY

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ABSTRACT. In this paper, we obtain some new inequalities for functions whose second derivatives' absolute value is s -convex and \log -convex. Also, we give some applications for numerical integration.

1. INTRODUCTION

We start with the well-known definition of convex functions: a function $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subset \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In the paper [12], authors gave the class of functions which are s -convex in the second sense by the following way. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the second sense if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in [0, \infty)$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. The class of s -convex functions in the second sense is usually denoted with K_s^2 .

Besides in [12], Hudzik and Maligranda proved that if $s \in (0, 1)$ $f \in K_s^2$ implies $f([0, \infty)) \subseteq [0, \infty)$, i.e., they proved that all functions from K_s^2 , $s \in (0, 1)$, are nonnegative.

Example 1. ([12]) Let $s \in (0, 1)$ and $a, b, c \in \mathbb{R}$. We define function $f : [0, \infty) \rightarrow \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

It can be easily checked that

- (i) If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$,
- (ii) If $b > 0$ and $c < 0$, then $f \notin K_s^2$.

Several researchers studied on s -convex functions, some of them can be found in [12]-[17].

Another kind of convexity is \log -convexity that is mentioned in [6] by Niculescu as following.

1991 *Mathematics Subject Classification.* 26D10, 26A15, 26A16, 26A51.

Key words and phrases. Convex function, s -convex function, \log -convex function, Ostrowski inequality, Hölder inequality, power-mean inequality.

This study was supported by Ağrı İbrahim Çeçen University BAP with project number FEF.14.011.

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A positive function f is called log $-$ convex on a real interval $I = [a, b]$, if for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq f^\lambda(x) f^{1-\lambda}(y).$$

For recent results for log $-$ convex functions, we refer to readers [2]-[9].

Now, we give a motivated inequality for convex functions:

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right]$$

is known as Bullen's inequality for convex functions [8], p. 39.

We also consider the following useful inequality:

Let $f : I \subset [0, \infty] \rightarrow \mathbb{R}$ be a differentiable mapping on I° , the interior of the interval I , such that $f' \in L[a, b]$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [11]).

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right]$$

This inequality is well known in the literature as the Ostrowski inequality.

The main aim of this paper is to prove some new integral inequalities for s -convex and log $-$ convex functions by using the integral identity that is obtained by Sarıkaya and Set in [1]. We also give some applications to our results in numerical integration. Some of our results are similar to the Ostrowski inequality and for special selections of the parameters, we proved some new inequalities of Bullen's type.

2. INEQUALITIES FOR s -CONVEX FUNCTIONS

We need the following Lemma which is obtained by Sarıkaya and Set in [1], so as to prove our results:

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping. Denote by $K(x, \cdot) : [a, b] \rightarrow \mathbb{R}$ the kernel given by*

$$K(x, t) = \begin{cases} \frac{\alpha}{\alpha+\beta} \frac{(t-a)(x-t)}{x-a}, & t \in [a, x] \\ -\frac{\beta}{\alpha+\beta} \frac{(b-t)(x-t)}{b-x}, & t \in [x, b] \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ nonnegative and not both zero, then the identity

$$\begin{aligned} & \int_a^b K(x, t) f''(t) dt \\ &= f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \end{aligned}$$

holds.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$,*

then

$$\begin{aligned} & \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\ & \leq \frac{\alpha}{\alpha + \beta} \frac{(x-a)^2}{(s+2)(s+3)} [|f''(x)| + |f''(a)|] \\ & \quad + \frac{\beta}{\alpha + \beta} \frac{(b-x)^2}{(s+2)(s+3)} [|f''(x)| + |f''(b)|] \end{aligned}$$

holds where $\alpha, \beta \in \mathbb{R}$ nonnegative and not both zero.

Proof. From Lemma 1, using the property of the modulus and s -convexity of $|f''|$, we can write

$$\begin{aligned} & \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\ & \leq \int_a^b |K(x, t)| |f''(t)| dt \\ & \leq \int_a^x \frac{\alpha}{\alpha + \beta} \frac{1}{x-a} |t-a| |x-t| |f''(t)| dt \\ & \quad + \int_x^b \frac{\beta}{\alpha + \beta} \frac{1}{b-x} |b-t| |x-t| |f''(t)| dt \\ & = \frac{\alpha}{(\alpha + \beta)(x-a)} \int_a^x (t-a)(x-t) \left| f'' \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right| dt \\ & \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \int_x^b (b-t)(t-x) \left| f'' \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right| dt \\ & \leq \frac{\alpha}{(\alpha + \beta)(x-a)} \int_a^x (t-a)(x-t) \left[\left(\frac{t-a}{x-a} \right)^s |f''(x)| + \left(\frac{x-t}{x-a} \right)^s |f''(a)| \right] dt \\ & \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \int_x^b (b-t)(t-x) \left[\left(\frac{t-x}{b-x} \right)^s |f''(b)| + \left(\frac{b-t}{b-x} \right)^s |f''(x)| \right] dt \\ & = \frac{\alpha}{\alpha + \beta} \frac{(x-a)^2}{(s+2)(s+3)} [|f''(x)| + |f''(a)|] + \frac{\beta}{\alpha + \beta} \frac{(b-x)^2}{(s+2)(s+3)} [|f''(x)| + |f''(b)|] \end{aligned}$$

where we use the fact that

$$\int_a^x (t-a)^{s+1} (x-t) dt = \int_a^x (t-a)(x-t)^{s+1} dt = \frac{(x-a)^{s+3}}{(s+2)(s+3)}$$

and

$$\int_x^b (b-t)(t-x)^{s+1} dt = \int_x^b (b-t)^{s+1} (t-x) dt = \frac{(b-x)^{s+3}}{(s+2)(s+3)}.$$

The proof is completed. \square

Corollary 1. *Suppose that all the assumptions of Theorem 1 are satisfied with $|f''| \leq M$. Then we have*

$$\begin{aligned} & \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\ & \leq \frac{2M}{(s+2)(s+3)} \left[\frac{\alpha(x-a)^2 + \beta(b-x)^2}{\alpha + \beta} \right]. \end{aligned}$$

Corollary 2. *In Theorem 1, if we choose $\alpha = \beta = 1$, we obtain*

$$\begin{aligned} & \left| f(x) + \frac{f(a) + f(b)}{2} - \left[\frac{1}{x-a} \int_a^x f(t) dt + \frac{1}{b-x} \int_x^b f(t) dt \right] \right| \\ & \leq \frac{(x-a)^2 + (b-x)^2}{2(s+2)(s+3)} |f''(x)| + \frac{1}{2(s+2)(s+3)} \left[(x-a)^2 |f''(a)| + (b-x)^2 |f''(b)| \right]. \end{aligned}$$

Corollary 3. *In Theorem 1, if we choose $\alpha = \beta = \frac{1}{2}$ and $x = \frac{a+b}{2}$, we obtain the following Bullen type inequality;*

$$\begin{aligned} & \left| \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a) + f(b)}{2} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)^2}{8(s+2)(s+3)} \left[\left| f''\left(\frac{a+b}{2}\right) \right| + \frac{|f''(a)| + |f''(b)|}{2} \right]. \end{aligned}$$

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|^q$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} & \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\ & \leq \left(\frac{\alpha}{\alpha + \beta} \right)^p \frac{(x-a)^{1+\frac{1}{q}}}{(s+1)^{\frac{1}{q}}} (\beta(p+1, p+1))^{\frac{1}{p}} [|f''(x)|^q + |f''(a)|^q]^{\frac{1}{q}} \\ & \quad + \left(\frac{\beta}{\alpha + \beta} \right)^p \frac{(b-x)^{1+\frac{1}{q}}}{(s+1)^{\frac{1}{q}}} (\beta(p+1, p+1))^{\frac{1}{p}} [|f''(b)|^q + |f''(x)|^q]^{\frac{1}{q}} \end{aligned}$$

where $\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$, $x, y > 0$ is the Euler Beta function, $\alpha, \beta \in \mathbb{R}$ nonnegative and not both zero.

Proof. From Lemma 1, using the property of the modulus, Hölder inequality and s -convexity of $|f''|^q$, we can write

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \left(\int_a^x \left(\frac{\alpha}{\alpha + \beta} \frac{(t-a)(x-t)^p}{x-a} \right)^p dt \right)^{\frac{1}{p}} \left(\int_a^x \left| f'' \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \quad + \left(\int_x^b \left(\frac{\beta}{\alpha + \beta} \frac{(b-t)(x-t)^p}{b-x} \right)^p dt \right)^{\frac{1}{p}} \left(\int_x^b \left| f'' \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\alpha}{\alpha + \beta} (x-a) \left(\int_a^x \frac{(t-a)^p (x-t)^p}{(x-a)^p (x-a)^p} dt \right)^{\frac{1}{p}} \left(\int_a^x \left| f'' \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta}{\alpha + \beta} (b-x) \left(\int_x^b \frac{(b-t)^p (x-t)^p}{(b-x)^p (b-x)^p} dt \right)^{\frac{1}{p}} \left(\int_x^b \left| f'' \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\alpha}{\alpha + \beta} (x-a) (\beta(p+1, p+1))^{\frac{1}{p}} \left[\int_a^x \left(\frac{t-a}{x-a} \right)^s |f''(x)|^q + \left(\frac{x-t}{x-a} \right)^s |f''(a)|^q \right]^{\frac{1}{q}} \\
& \quad + \frac{\beta}{\alpha + \beta} (b-x) (\beta(p+1, p+1))^{\frac{1}{p}} \left[\int_x^b \left(\frac{t-x}{b-x} \right)^s |f''(b)|^q + \left(\frac{b-t}{b-x} \right)^s |f''(x)|^q \right]^{\frac{1}{q}}.
\end{aligned}$$

We get the desired result by making use of the necessary computation. \square

Theorem 3. Under the assumptions of Theorem 2, the following inequality

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \beta(p+1, p+1)^{\frac{1}{p}} \left(\left(\frac{\alpha}{\alpha + \beta} \right)^p (x-a)^p + \left(\frac{\beta}{\alpha + \beta} \right)^p (b-x)^p \right)^{\frac{1}{p}} \\
& \quad \times \left(\frac{b-a}{s+1} \right)^{\frac{1}{q}} (|f''(a)|^q + |f''(b)|^q)^{\frac{1}{q}}
\end{aligned}$$

holds where $\beta(x, y)$ is the Euler Beta function.

Proof. From Lemma 1, using the property of the modulus, Hölder inequality and s -convexity of $|f''|^q$, we can write

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \left(\int_a^b |K(x, t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f''(t)|^q dt \right)^{\frac{1}{q}} \\
& = \left(\int_a^x \left(\frac{\alpha}{\alpha + \beta} \frac{(t-a)(x-t)}{(x-a)(x-a)} (x-a) \right)^p dt + \int_x^b \left(\frac{\beta}{\alpha + \beta} \frac{(b-t)(x-t)}{(b-x)(b-x)} (b-x) \right)^p dt \right)^{\frac{1}{p}} \\
& \quad \times \left(\int_a^b \left| f'' \left(\frac{t-a}{b-a} b + \frac{b-t}{b-a} a \right) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

We get the desired result by making use of the necessary computation. □

The next result is obtained by using the well-known power-mean integral inequality:

Theorem 4. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|^q$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then*

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \frac{\alpha}{\alpha + \beta} \frac{(x-a)^2}{6^{1-\frac{1}{q}} [(s+2)(s+3)]^{\frac{1}{q}}} [|f''(x)|^q + |f''(a)|^q]^{\frac{1}{q}} \\
& \quad + \frac{\beta}{\alpha + \beta} \frac{(b-x)^2}{6^{1-\frac{1}{q}} [(s+2)(s+3)]^{\frac{1}{q}}} [|f''(b)|^q + |f''(x)|^q]^{\frac{1}{q}}
\end{aligned}$$

holds where $\alpha, \beta \in \mathbb{R}$ nonnegative and not both zero.

Proof. From Lemma 1, using the property of the modulus, power-mean integral inequality and s -convexity of $|f''|^q$, we can write

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \frac{\alpha}{(\alpha + \beta)(x-a)} \left(\int_a^x (t-a)(x-t) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_a^x (t-a)(x-t) \left(\left(\frac{t-a}{x-a} \right)^s |f''(x)|^q + \left(\frac{x-t}{x-a} \right)^s |f''(a)|^q \right) dt \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \left(\int_x^b (b-t)(t-x) dt \right)^{1-\frac{1}{q}} \\
& \quad \times \left(\int_x^b (t-b)(t-x) \left(\left(\frac{t-x}{b-x} \right)^s |f''(b)|^q + \left(\frac{b-t}{b-x} \right)^s |f''(x)|^q \right) dt \right)^{\frac{1}{q}} \\
& = \frac{\alpha}{(\alpha + \beta)(x-a)} \left(\frac{(x-a)^3}{6} \right)^{1-\frac{1}{q}} \left(\frac{(x-a)^3}{(s+2)(s+3)} (|f''(x)|^q + |f''(a)|^q) \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \left(\frac{(b-x)^3}{6} \right)^{1-\frac{1}{q}} \left(\frac{(b-x)^3}{(s+2)(s+3)} (|f''(b)|^q + |f''(x)|^q) \right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is completed. \square

Remark 1. In Theorem 4, if we choose $q = 1$ Theorem 4 reduces to Theorem 1.

Remark 2. If we choose $s = 1$ for all the results, we obtain new results for convex functions.

3. INEQUALITIES FOR \log -CONVEX FUNCTIONS

In this section, we will give some results for \log -convex functions. For the simplicity, we will use the following notations:

$$\begin{aligned}
\kappa &= \left(\frac{|f''(x)|}{|f''(a)|} \right)^{\frac{1}{x-a}} \\
\tau &= \left(\frac{|f''(b)|}{|f''(x)|} \right)^{\frac{1}{b-x}}.
\end{aligned}$$

Theorem 5. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|$ is \log -convex function on $[a, b]$ and $\kappa \neq 1, \tau \neq 1$, then

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \frac{\alpha}{(\alpha + \beta)(x-a)} \left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \left(\frac{2(\kappa^a - \kappa^x) - (a-x)(\kappa^a + \kappa^x) \log \kappa}{\log^3 \kappa} \right) \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \left(\frac{2\tau^x - 2\tau^b + (b-x)(\tau^b + \tau^x) \log \tau}{\log^3 \tau} \right)
\end{aligned}$$

holds where $\kappa \neq 1, \tau \neq 1, \alpha, \beta \in \mathbb{R}$ nonnegative and not both zero.

Proof. From Lemma 1 and by using the log – convexity of $|f''|$, we can write

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \int_a^x \frac{\alpha}{\alpha + \beta} \frac{1}{x-a} |t-a| |x-t| |f''(t)| dt \\
& \quad + \int_x^b \frac{\beta}{\alpha + \beta} \frac{1}{b-x} |b-t| |x-t| |f''(t)| dt \\
& = \frac{\alpha}{(\alpha + \beta)(x-a)} \int_a^x (t-a)(x-t) \left| f'' \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right| dt \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \int_x^b (b-t)(t-x) \left| f'' \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right| dt \\
& \leq \frac{\alpha}{(\alpha + \beta)(x-a)} \int_a^x (t-a)(x-t) \left[|f''(x)|^{\frac{t-a}{x-a}} |f''(a)|^{\frac{x-t}{x-a}} \right] dt \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \int_x^b (b-t)(t-x) \left[|f''(b)|^{\frac{t-x}{b-x}} |f''(x)|^{\frac{b-t}{b-x}} \right] dt \\
& = \frac{\alpha}{(\alpha + \beta)(x-a)} \left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \int_a^x (t-a)(x-t) \kappa^t dt \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \int_x^b (b-t)(t-x) \tau^t dt.
\end{aligned}$$

By a simple computation, we get the result. □

Corollary 4. In Theorem 5, if we choose $\alpha = \beta = 1$, we obtain

$$\begin{aligned}
& \left| f(x) + \frac{f(a) + f(b)}{2} - \left[\frac{1}{x-a} \int_a^x f(t) dt + \frac{1}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \frac{1}{2(x-a)} \left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \left(\frac{2(\kappa^a - \kappa^x) - (a-x)(\kappa^a + \kappa^x) \log \kappa}{\log^3 \kappa} \right) \\
& \quad + \frac{1}{2(b-x)} \left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \left(\frac{2\tau^x - 2\tau^b + (b-x)(\tau^b + \tau^x) \log \tau}{\log^3 \tau} \right).
\end{aligned}$$

Corollary 5. *In Theorem 5, if we choose $\alpha = \beta = \frac{1}{2}$ and $x = \frac{a+b}{2}$, we obtain the following Bullen type inequality;*

$$\begin{aligned} & \left| \frac{1}{2} \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} \right] - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left(\frac{|f''(a)|^{\frac{a+b}{2}}}{|f''(\frac{a+b}{2})|^a} \right)^{\frac{2}{b-a}} \left(\frac{\kappa_1^a - \kappa_1^{\frac{a+b}{2}} + (\frac{b-a}{4}) (\kappa_1^a + \kappa_1^{\frac{a+b}{2}}) \log \kappa_1}{(b-a) \log^3 \kappa_1} \right) \\ & \quad + \left(\frac{|f''(\frac{a+b}{2})|^b}{|f''(b)|^{\frac{a+b}{2}}} \right)^{\frac{2}{b-a}} \left(\frac{\tau_1^{\frac{a+b}{2}} - \tau_1^b + (\frac{b-a}{4}) (\tau_1^b + \tau_1^{\frac{a+b}{2}}) \log \tau_1}{(b-a) \log^3 \tau_1} \right) \end{aligned}$$

where

$$\begin{aligned} \kappa_1 &= \left(\frac{|f''(\frac{a+b}{2})|}{|f''(a)|} \right)^{\frac{2}{b-a}} \\ \tau_1 &= \left(\frac{|f''(b)|}{|f''(\frac{a+b}{2})|} \right)^{\frac{2}{b-a}}. \end{aligned}$$

Theorem 6. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|^q$ is \log -convex function on $[a, b]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} & \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\ & \leq \frac{\alpha}{\alpha + \beta} (x-a) (\beta(p+1, p+1))^{\frac{1}{p}} \left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \left(\frac{\kappa^{\frac{qx}{x-a}} - \kappa^{\frac{qa}{x-a}}}{\log \kappa^{\frac{q}{x-a}}} \right)^{\frac{1}{q}} \\ & \quad + \frac{\beta}{\alpha + \beta} (b-x) (\beta(p+1, p+1))^{\frac{1}{p}} \left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \left(\frac{\tau^{\frac{qb}{b-x}} - \tau^{\frac{qx}{b-x}}}{\log \tau^{\frac{q}{b-x}}} \right)^{\frac{1}{q}}. \end{aligned}$$

where $\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$, $x, y > 0$ is the Euler Beta function and $\kappa \neq 1, \tau \neq 1, \alpha, \beta \in \mathbb{R}$ nonnegative and not both zero.

Proof. From Lemma 1, by using log-convexity of $|f''|^q$ and by applying Hölder inequality, we get

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \left(\int_a^x \left(\frac{\alpha}{\alpha + \beta} \frac{(t-a)(x-t)}{x-a} \right)^p dt \right)^{\frac{1}{p}} \left(\int_a^x \left| f'' \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \quad + \left(\int_x^b \left(\frac{\beta}{\alpha + \beta} \frac{(b-t)(x-t)}{b-x} \right)^p dt \right)^{\frac{1}{p}} \left(\int_x^b \left| f'' \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\alpha}{\alpha + \beta} (x-a) \left(\int_a^x \frac{(t-a)^p (x-t)^p}{(x-a)^p (x-a)^p} dt \right)^{\frac{1}{p}} \left(\int_a^x \left| f'' \left(\frac{t-a}{x-a} x + \frac{x-t}{x-a} a \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta}{\alpha + \beta} (b-x) \left(\int_x^b \frac{(b-t)^p (x-t)^p}{(b-x)^p (b-x)^p} dt \right)^{\frac{1}{p}} \left(\int_x^b \left| f'' \left(\frac{t-x}{b-x} b + \frac{b-t}{b-x} x \right) \right|^q dt \right)^{\frac{1}{q}} \\
& \leq \frac{\alpha}{\alpha + \beta} (x-a) (\beta(p+1, p+1))^{\frac{1}{p}} \left[\left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{q}{x-a}} \int_a^x \left(\frac{|f''(x)|^{\frac{q}{x-a}}}{|f''(a)|^{\frac{q}{x-a}}} \right)^t dt \right]^{\frac{1}{q}} \\
& \quad + \frac{\beta}{\alpha + \beta} (b-x) (\beta(p+1, p+1))^{\frac{1}{p}} \left[\left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{q}{b-x}} \int_x^b \left(\frac{|f''(b)|^{\frac{q}{b-x}}}{|f''(x)|^{\frac{q}{b-x}}} \right)^t dt \right]^{\frac{1}{q}}.
\end{aligned}$$

By computing the above integrals, we get the desired result. \square

Theorem 7. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|^q$ is log-convex function on $[a, b]$ and $q \geq 1$, then

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \frac{\alpha (x-a)^{2-\frac{3}{q}}}{6^{1-\frac{1}{q}} (\alpha + \beta)} \left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \left(\frac{2(\kappa^{qa} - \kappa^{qx}) - q(a-x)(\kappa^{qa} + \kappa^{qx}) \log \kappa}{\log^3 \kappa^q} \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta (b-x)^{2-\frac{3}{q}}}{6^{1-\frac{1}{q}} (\alpha + \beta)} \left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \left(\frac{2\tau^{qx} - 2\tau^{qb} + q(b-x)(\tau^{qb} + \tau^{qx}) \log \tau}{\log^3 \tau^q} \right)^{\frac{1}{q}}
\end{aligned}$$

holds where $\kappa^q \neq 1$, $\tau^q \neq 1$, $\alpha, \beta \in \mathbb{R}$ nonnegative and not both zero.

Proof. From Lemma 1, by using the well-known power-mean integral inequality and \log -convexity of $|f''|^q$, we have

$$\begin{aligned}
& \left| f(x) + \frac{\alpha f(a) + \beta f(b)}{\alpha + \beta} - \frac{2}{\alpha + \beta} \left[\frac{\alpha}{x-a} \int_a^x f(t) dt + \frac{\beta}{b-x} \int_x^b f(t) dt \right] \right| \\
& \leq \frac{\alpha}{(\alpha + \beta)(x-a)} \left(\int_a^x (t-a)(x-t) dt \right)^{1-\frac{1}{q}} \left(\int_a^x (t-a)(x-t) \left(|f''(x)|^{q\frac{t-a}{x-a}} |f''(a)|^{q\frac{x-t}{x-a}} \right) dt \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \left(\int_x^b (b-t)(t-x) dt \right)^{1-\frac{1}{q}} \left(\int_x^b (t-b)(t-x) \left(|f''(b)|^{q\frac{t-x}{b-x}} |f''(x)|^{q\frac{b-t}{b-x}} \right) dt \right)^{\frac{1}{q}} \\
& = \frac{\alpha}{(\alpha + \beta)(x-a)} \left(\frac{(x-a)^3}{6} \right)^{1-\frac{1}{q}} \left(\frac{|f''(a)|^x}{|f''(x)|^a} \right)^{\frac{1}{x-a}} \left(\frac{2(\kappa^{qa} - \kappa^{qx}) - q(a-x)(\kappa^{qa} + \kappa^{qx}) \log \kappa}{\log^3 \kappa^q} \right)^{\frac{1}{q}} \\
& \quad + \frac{\beta}{(\alpha + \beta)(b-x)} \left(\frac{(b-x)^3}{6} \right)^{1-\frac{1}{q}} \left(\frac{|f''(x)|^b}{|f''(b)|^x} \right)^{\frac{1}{b-x}} \left(\frac{2\tau^{qx} - 2\tau^{qb} + q(b-x)(\tau^{qb} + \tau^{qx}) \log \tau}{\log^3 \tau^q} \right)^{\frac{1}{q}}.
\end{aligned}$$

Which completes the proof. \square

Remark 3. In Theorem 7, if we choose $q = 1$ Theorem 7 reduces to Theorem 5.

Corollary 6. For the particular selections of the parameters α, β and the variable x , one can obtain several new inequalities for \log -convex functions, we omit the details.

4. APPLICATIONS FOR NUMERICAL INTEGRATION

Suppose that $d = \{a = x_0 < x_1 < \dots < x_n = b\}$ is a partition of the interval $[a, b]$, $h_i = x_{i+1} - x_i$, for $i = 0, 1, 2, \dots, n-1$ and consider the averaged midpoint-trapezoid quadrature formula

$$\int_a^b f(x) dx = A_{MT}(d, f) + R_{MT}(d, f),$$

where

$$A_{MT}(\pi, f) = \frac{1}{4} \sum_{i=0}^{n-1} h_i \left[f(x_i) + 2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1}) \right]$$

Here, the term $R_{MT}(d, f)$ denotes the associated approximation error. (See [10])

Proposition 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|$ is \log -convex function on $[a, b]$ and $\kappa_1 \neq 1, \tau_1 \neq 1$, then for the partition d , following inequality holds

$$\begin{aligned}
& |R_{MT}(d, f)| \\
& \leq \left(\frac{|f''(x_i)|^{\frac{x_i + x_{i+1}}{2}}}{|f''(\frac{x_i + x_{i+1}}{2})|^{x_i}} \right)^{\frac{2}{h_i}} \left(\frac{\kappa_i^{x_i} - \kappa_i^{\frac{x_i + x_{i+1}}{2}} + (\frac{h_i}{4}) \left(\kappa_i^{x_i} + \kappa_i^{\frac{x_i + x_{i+1}}{2}} \right) \log \kappa_i}{h_i \log^3 \kappa_i} \right) \\
& \quad + \left(\frac{|f''(\frac{x_i + x_{i+1}}{2})|^{x_{i+1}}}{|f''(x_{i+1})|^{\frac{x_i + x_{i+1}}{2}}} \right)^{\frac{2}{h_i}} \left(\frac{\tau_i^{\frac{x_i + x_{i+1}}{2}} - \tau_i^{x_{i+1}} + (\frac{h_i}{4}) \left(\tau_i^{x_{i+1}} + \tau_i^{\frac{x_i + x_{i+1}}{2}} \right) \log \tau_i}{h_i \log^3 \tau_i} \right).
\end{aligned}$$

where $\kappa_i \neq 1, \tau_i \neq 1$ and defined as

$$\begin{aligned}\kappa_i &= \left(\frac{\left| f'' \left(\frac{x_i + x_{i+1}}{2} \right) \right|}{|f''(x_i)|} \right)^{\frac{2}{h_i}} \\ \tau_i &= \left(\frac{|f''(x_{i+1})|}{\left| f'' \left(\frac{x_i + x_{i+1}}{2} \right) \right|} \right)^{\frac{2}{h_i}}.\end{aligned}$$

Proof. By applying Corollary 5 to the subintervals $[x_i, x_{i+1}]$ of d , ($i = 0, 1, \dots, n-1$) and by summation. We obtain the desired result. \square

Proposition 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping such that $f'' \in L[a, b]$. If $|f''|$ is s -convex in the second sense on $[a, b]$ for some fixed $s \in (0, 1]$, then for partition d of $[a, b]$ the following inequality holds:

$$\begin{aligned}& |R_{MT}(d, f)| \\ & \leq \frac{h_i^2}{8(s+2)(s+3)} \left[\left| f'' \left(\frac{x_i + x_{i+1}}{2} \right) \right| + \frac{|f''(x_i)| + |f''(x_{i+1})|}{2} \right].\end{aligned}$$

Proof. By applying Corollary 3 to the subintervals $[x_i, x_{i+1}]$ of d , ($i = 0, 1, \dots, n-1$) and by summation. we get the result. \square

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